technische universität
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Image Reconstruction

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Detector systems in particle and medical physics

Outline

Basic concepts

Analytic reconstruction algorithms

3D image reconstruction

Algebraic reconstruction

Summary

Reference: *Zeng, G. (2023). Medical Image Reconstruction: From Analytical and Iterative Methods to Machine Learning. Berlin, Boston: De Gruyter.*

Basic concepts

Definition

Image reconstruction is the procedure to produce a **tomographic image** from **projections**.

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- Tomographic image

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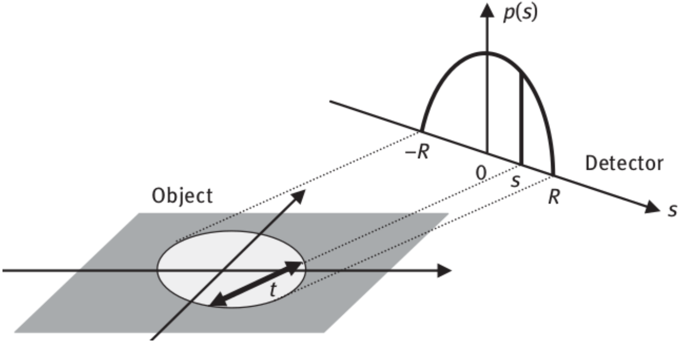
- Projection



Sinogram

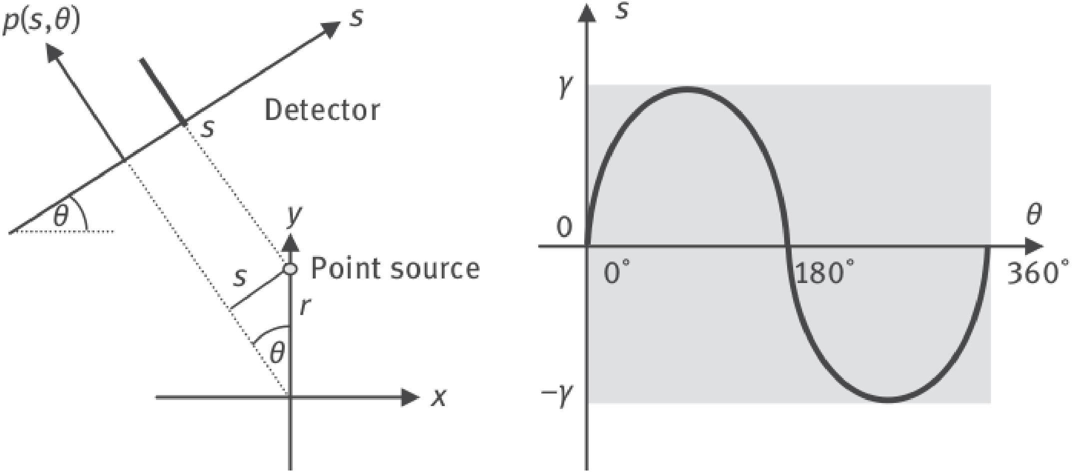
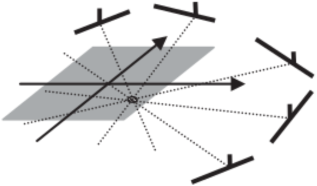


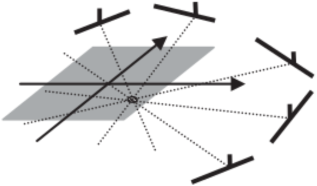
Figure: A sinogram is a representation of the projections on the s-θ plane.

Backprojection

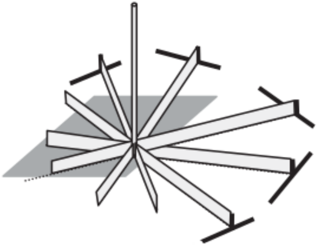


(a)

Backprojection



(*a*)



(*b*)

Backprojection

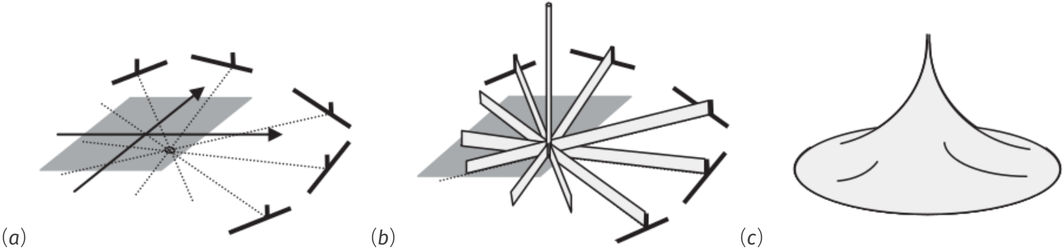


Figure: ”Reconstruction” of a point source. In (*a*) some projections are taken from different positions. (*b*) shows the obtained backprojection using only those positions. (*c*) shows the obtained shape after backprojecting from all positions.

Backprojection

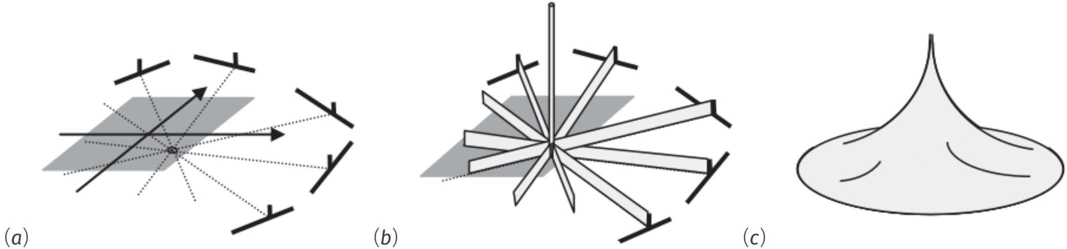
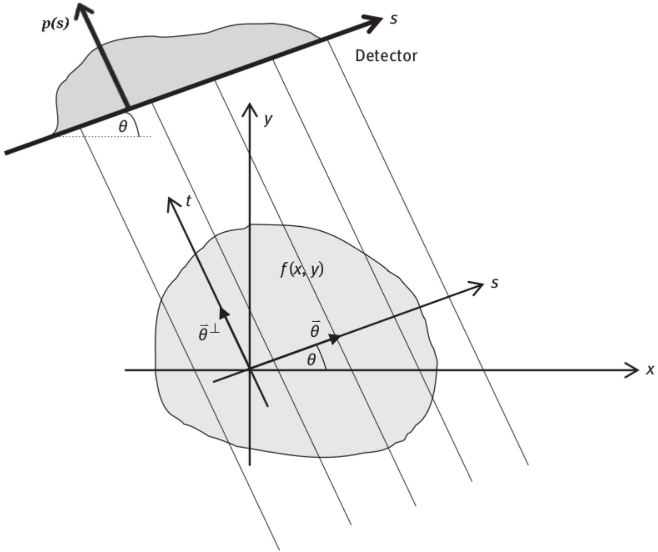


Figure: ”Reconstruction” of a point source. In (*a*) some projections are taken from different positions. (*b*) shows the obtained backprojection using only those positions. (*c*) shows the obtained shape after backprojecting from all positions.

Backprojection does most of the work, but we need some algorithm to reconstruct the original function.

Mathematical definitions



- Projection

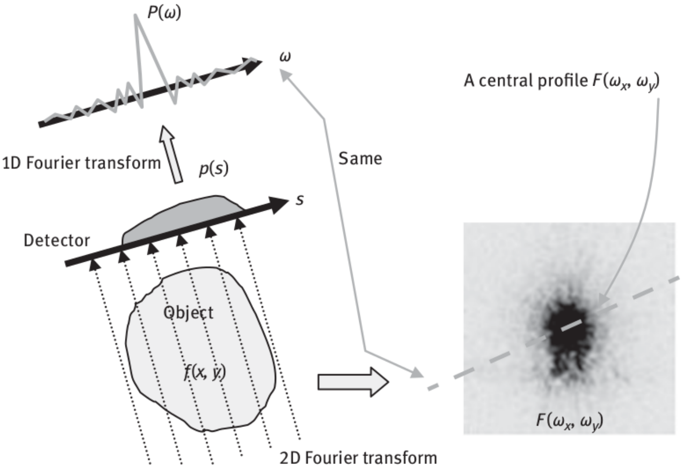
$$p(s,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta + y\sin\theta - s)\,dx\,dy$$

- Backprojection

$$b(x,y) = \int_0^{\pi} p(s,\theta)\big|_{s=x\cos\theta+y\sin\theta}\,d\theta$$

Analytic reconstruction algorithms

Central Slice Theorem (CST)



$$P(\omega, \theta) = F(\omega \cos\theta, \omega \sin\theta)$$

The 1D Fourier transform $P(\omega)$ of the projection $p(s)$ of a 2D function $f(x, y)$ is equal to a slice (i.e., a 1D profile) through the origin of the 2D Fourier transform $F(\omega_x, \omega_y)$ of that function which is parallel to the detector.

Filtering

Over-weighting with low-frequency components blurs the image. This effect can be compensated in the Fourier space.

- Multiply the $\omega_x - \omega_y$ space Fourier "image" by $\sqrt{\omega_x^2 + \omega_y^2}$.
- Multiply the 1D Fourier transform $P(\omega, \theta)$ of the projection data $p(s, \theta)$ by $|\omega|$.

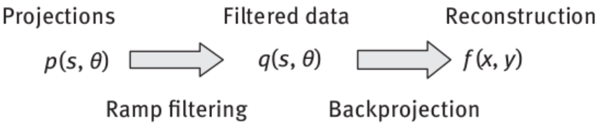


Figure: The procedure of the filtered backprojection (FBP) algorithm.

FBP algorithm

Starting with the 2D inverse Fourier transform in polar coordinates:

$$f(x,y) = \int_0^{2\pi}\int_0^{\infty} F_{\text{polar}}(\omega,\theta)e^{2\pi i\omega(x\cos\theta+y\sin\theta)}\omega d\omega d\theta \quad ; \qquad F_{polar}(\omega,\theta) = F_{polar}(-\omega,\theta+\pi)$$

$$\implies f(x,y) = \int_0^{\pi}\int_{-\infty}^{\infty} F_{\text{polar}}(\omega,\theta)|\omega|e^{2\pi i\omega(x\cos\theta+y\sin\theta)}d\omega d\theta$$

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By using the central slice theorem, we can replace F by P :

$$f(x,y) = \int_0^{\pi} \int_{-\infty}^{\infty} P(\omega,\theta) |\omega| e^{2\pi i \omega (x\cos\theta + y\sin\theta)} d\omega d\theta.$$

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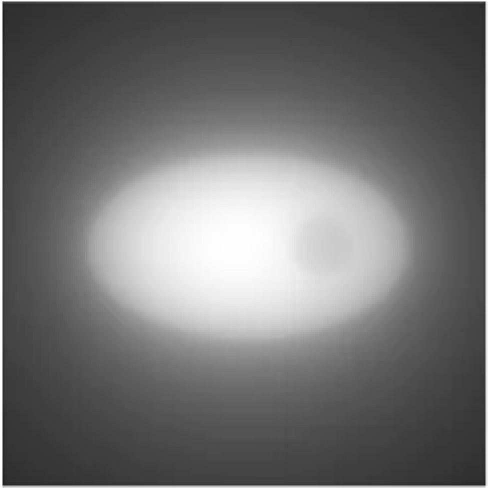
Let $Q(\omega,\theta) = |\omega|P(\omega,\theta)$, then

$$f(x,y) = \int_0^{\pi}\int_{-\infty}^{\infty} Q(\omega,\theta)e^{2\pi i\omega(x\cos\theta+y\sin\theta)}d\omega d\theta.$$

$$f(x,y) = \int_0^{\pi} q(x\cos\theta + y\sin\theta,\theta)d\theta = \int_0^{\pi} q(s,\theta)\Big|_{s=x\cos\theta+y\sin\theta} d\theta.$$

Backprojection vs Filtered Backprojection



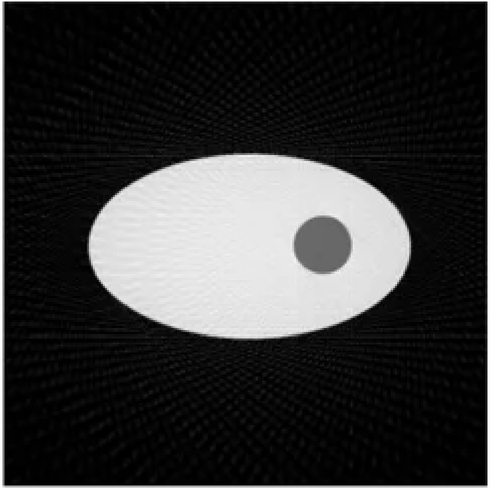


Figure: Reconstruction using 120 projections without ramp filter $|\omega|$ (left) and with ramp filter (right).

Fourier transform properties

Fourier transform properties

- Multiplication in one domain corresponds to convolution in the other domain. The convolution of two functions f and g is defined as

$$(f * g) = (t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

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→ These properties can be used to create new reconstruction algorithms.

Other filtering techniques

Other filtering techniques

- **Method 1**: FBP algorithm.

- **Method 2**: The ramp-filtered data $q(s, \theta)$ can be obtained by convolution as:

$$q(s,\theta) = p(s,\theta) * h(s),$$

Here $h(s)$ is the convolution kernel and is the 1D inverse Fourier transform of $H(\omega) = |\omega|$.

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- **Method 3**: Let's factor the ramp filter into two parts:

$$H(\omega) = |\omega| = i2\pi\omega \cdot \frac{1}{i2\pi} sgn(\omega) = i2\pi\omega \cdot \frac{-i}{2\pi} sgn(\omega)$$

$$\implies q(s, \theta) = \frac{dp(s, \theta)}{ds} * \frac{-1}{2\pi^2 s}$$

Other filtering techniques

- **Method 4**: Switch the order of ramp filtering and backprojection:

  (i) Find the 2D Fourier transform of the blurred image obtained after backprojection $b(x,y)$, obtaining $B(\omega_x, \omega_y)$.

  (ii) Multiply $B\left(\omega_x, \omega_y\right)$ with a ramp filter $|\omega| = \sqrt{\omega_x^2 + \omega_y^2}$, obtaining $F\left(\omega_x, \omega_y\right)$.

  (iii) Find the 2D inverse Fourier transform of $F\left(\omega_x, \omega_y\right)$, obtaining $f(x,y)$.

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  (iii) Find the 2D inverse Fourier transform of $F\left(\omega_x, \omega_y\right)$, obtaining $f(x,y)$.

→ All the previous methods provide an exact reconstruction of the function $f(x,y)$.

Fan-beam reconstruction

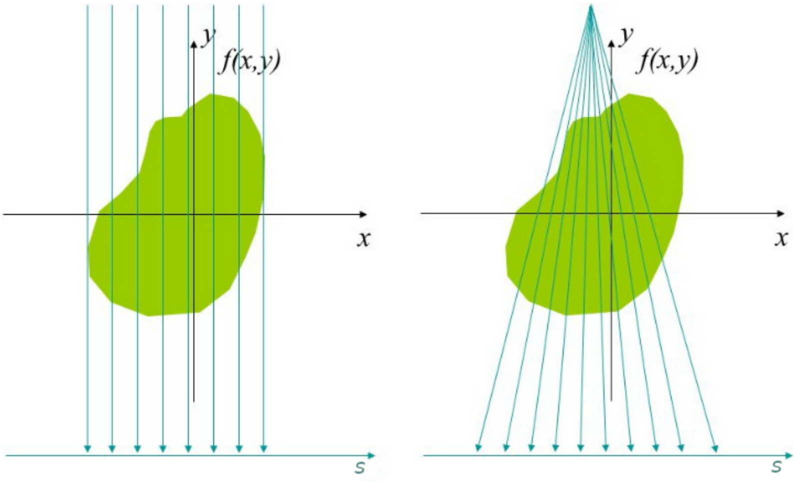
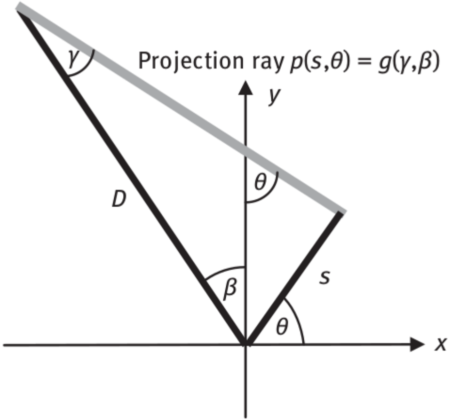


Figure: Comparison between parallel and fan beam reconstruction.

Fan-beam geometry



- $\theta = \gamma + \beta$
- $s = D \sin \gamma$

A fan-beam ray can be represented using the parallel-beam geometry parameters.

Fan-beam algorithms

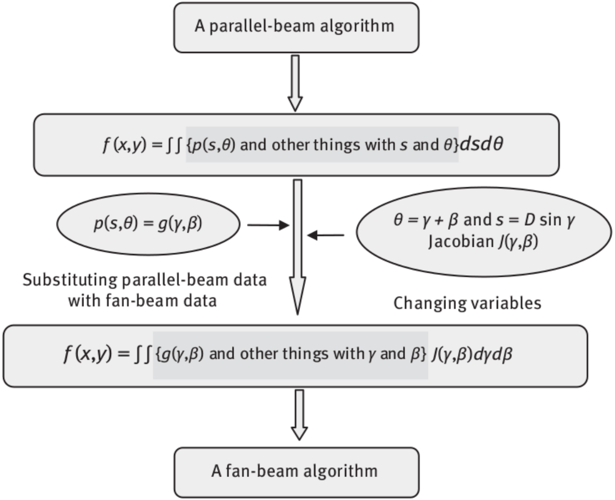


Figure: The procedure to change a parallel-beam algorithm into a fan-beam algorithm.

Peculiarities of fan-beam reconstruction

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  - → Derivative + Hilbert transform algorithm gets rid of this problem.
- Redundant measurements even when short scanning.
  - → Proper weighting during image reconstruction is needed.
- In real implementation, the integral $h(\gamma) = \int_{-\infty}^{\infty} |\omega| e^{i2\pi\omega\gamma} d\omega$ (inverse Fourier transform of $|\omega|$) is not performed from $-\infty$ to $+\infty$, but a finite bandwidth is used. The uncertainties coming from this step can become important when short scan is employed.

3D image reconstruction

Parallel line-integral data

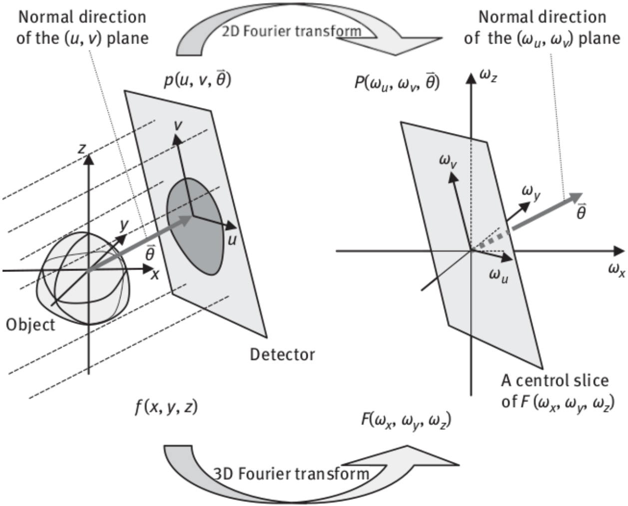
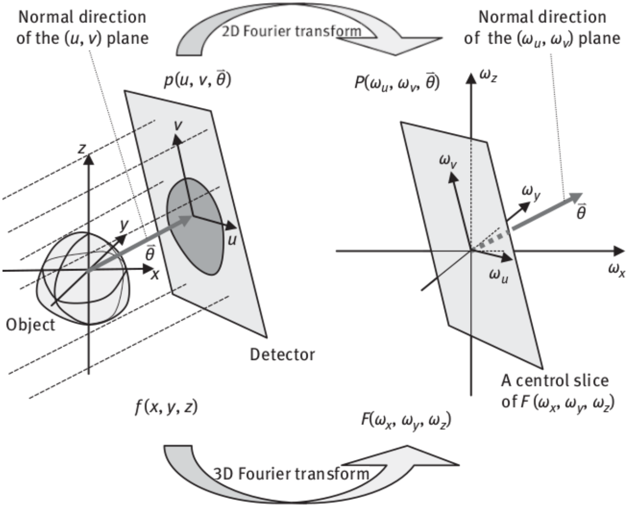
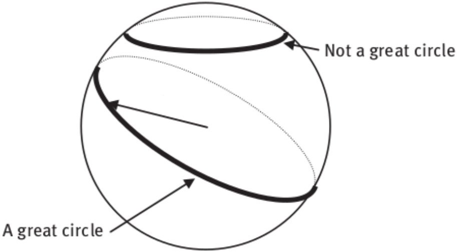


Figure: Central slice theorem for the 3D line-integral projections.

Parallel line-integral data





If $\vec{\theta}$ follows the trajectory of a **great circle**, the $(\omega_x, \omega_y, \omega_z)$ Fourier space is filled up.

Figure: Central slice theorem for the 3D line-integral projections.

Cone-beam data

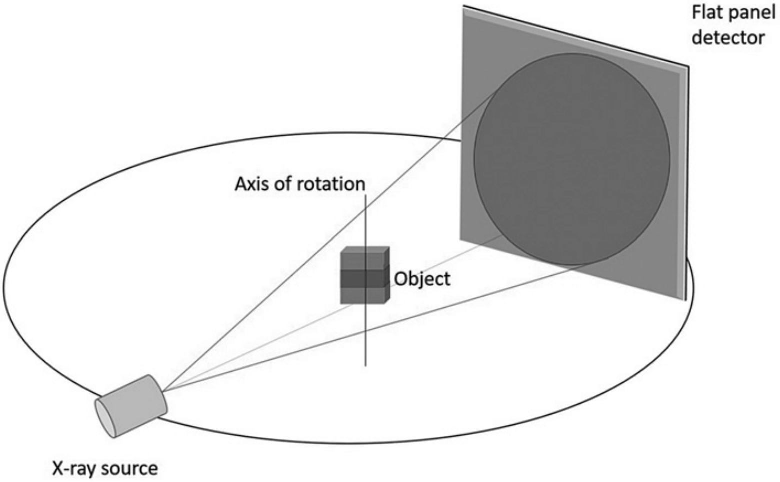


Figure: Cone beam data acquisition. There is no equivalent Central Slice Theorem.

Algebraic reconstruction

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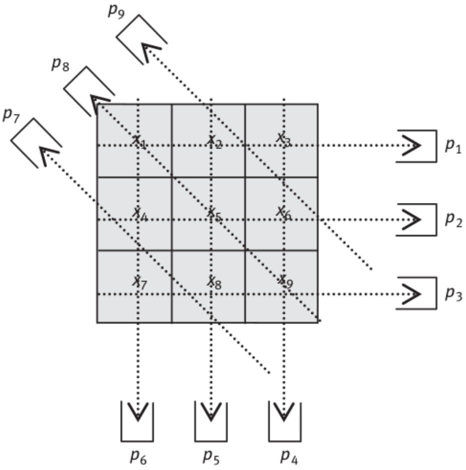


System of linear equations written in matrix form:

$$AX = P$$

where

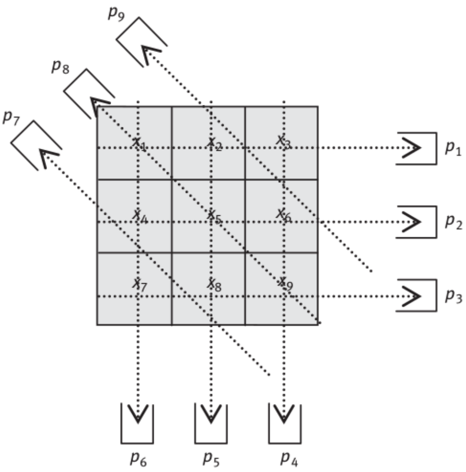
$X = [x_1, x_2, ..., x_9]^T$

$P = [p_1, p_2, ..., p_9]^T$

$A \equiv$ *weighting matrix*

Figure: An example with nine unknowns and nine measurements.

Algebraic reconstruction



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Inverting to obtain the desired matrix X:

$$X = A^{-1}P$$

In general, calculate A^{-1} is not an easy task.

Iterative reconstruction

Usually the matrix A is too large to be stored in a computer, so it is generated one row at a time.

$\implies$ Iterative methods that only use A and A^T make sense in finding an approximate solution.

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- Least-squares minimisation: $\chi^2 = |AX - P|^2$
  - Use of singular value decomposition (SVD) to find a pseudo-inverse.
  - Gradient descent.

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- Least-squares minimisation: $\chi^2 = |AX - P|^2$
  - Use of singular value decomposition (SVD) to find a pseudo-inverse.
  - Gradient descent.
- Maximise the likelihood of the probability density function associated with the noise in the projections.
  A Poisson distribution can be assumed.

FBP vs Iterative reconstruction

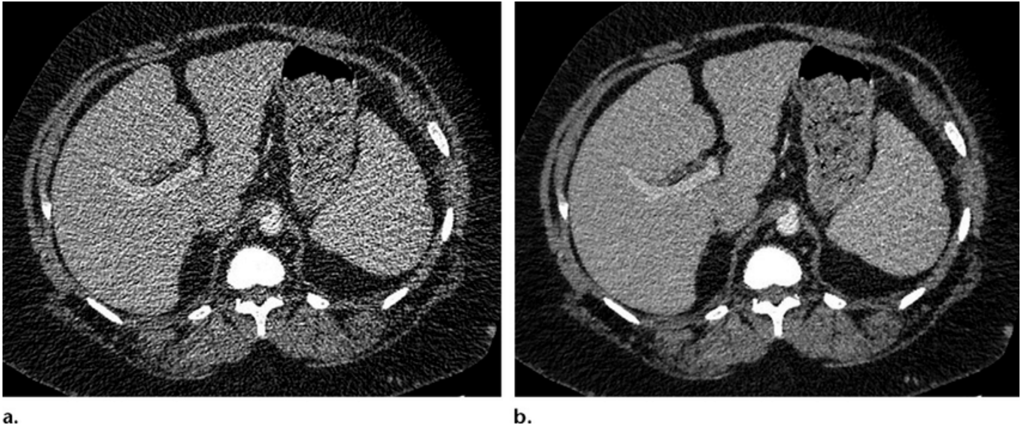

Figure: Comparison between the reconstruction obtained by filtered backprojection **(a)** and iterative reconstruction **(b)**.
Source: D. Fursevich *et al.*, "Bariatric CT Imaging: Challenges and Solutions".

How to measure the projections

How to measure the projections

- Beer's law

The attenuation of X-rays along a line of material can be modeled using the Beer's law.

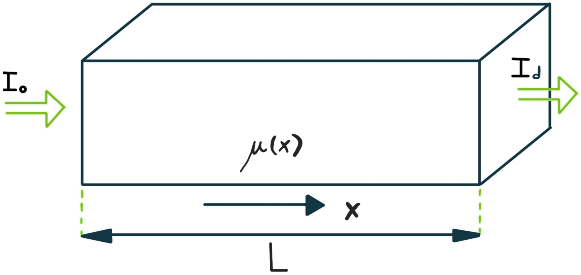

Figure: Attenuation along a line.

$$I_d = I_0 e^{-\int_L \mu(x)dx} = I_0 e^{-p}$$

$$p = ln\left(\frac{I_0}{I_d}\right)$$

Summary

Summary

- Projection and backprojection are the fundamental concepts in image reconstruction.
- The Central Slice Theorem is a key tool to develop analytic image reconstruction algorithms.
- The algorithms must be adapted for the different geometries: parallel or fan beams.
- 3D image reconstruction can be achieved based on the same principles.
- Lately, iterative image reconstruction algorithms are getting more and more attention in medical image reconstruction.
- The projections are obtained by measuring the attenuation of the radiation through the body.

Appendix: Central Slice Theorem demonstration

We start with the definition of the 1D Fourier transform:

$$P(\omega) = \int_{-\infty}^{\infty} p(s) e^{-2\pi i s \omega} ds,$$

then use the definition of $p(s, \theta)$, obtaining

$$P(\omega, \theta) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos\theta + y \sin\theta - s) dx dy \right] e^{-2\pi i s \omega} ds.$$

Changing the order of integrals yields

$$P(\omega, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \left[\int_{-\infty}^{\infty} \delta(x \cos\theta + y \sin\theta - s) e^{-2\pi i s \omega} ds \right] dx dy.$$

Using the property of the δ function, the inner integral over s can be readily obtained and we have

$$P(\omega, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-2\pi i (x \cos\theta + y \sin\theta)\omega} dx dy,$$

that is,

$$P(\omega, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-2\pi i (xu + yv)} \Big|_{u = \omega\cos\theta, v = \omega\sin\theta} dx dy.$$

Finally, using the definition of the 2D Fourier transform yields

$$P(\omega, \theta) = F\left(\omega_x, \omega_y\right)\Big|_{\omega_x = \omega\cos\theta, \omega_y = \omega\sin\theta}.$$